Mathematics of Computing
Final Exam (Max Marks 50, Time 3h)

Indian Statistical Institute, Bangalore

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NB: (a) Here w^r represents the reverse of the string w.

Q1. $(2 \times 5 = 10)$ In the following, assume that regular languages are closed under the operations Union, Complement and Star.

(a) Prove or disprove: The intersection of two regular languages is regular.

(b) Prove or disprove: If L and M are regular languages then the difference $L - M$ is also regular.

(c) Let L be a regular language. Let $L^r = \{w^r | w \in L\}$. Show that the language L^r is regular.

(d) Let L be a regular language. Let L^s be a language where every occurrence of the symbol a in every string $w \in L$ is replaced by the symbol b. Note that b may also occur in w. Show that L^s is regular.

(e) Show that $L = \{0^n | n \text{ is prime }\}$ is not regular.

Q2. $(4 + 2 + 4 = 10)$

(a) Consider the language $L = \{ww^r | w \text{ is a binary string }\}$. Prove or disprove that L is context free.

(b) State clearly the *pumping lemma* for context free languages.

(c) Prove that the language $L = \{ww | w \text{ is a binary string}\}$ is not context free.

Q3. $(2 \times 6 = 12)$ State if true or false and then prove your claim. You may assume, as proved in class, that A_{TM} is Turing recognizable but not Turing decidable.

(a) The language $\{< D, w > | D \text{ is a DFA and } D \text{ accepts } w\}$ is Turing decidable.

(b) The language $\{< G, w > | G \text{ is a CFG and } w \in L(G)\}$ is Turing decidable.

(c) The language $\overline{A_{TM}}$ is Turing recognizable.

(d) It is possible to list all languages over strings over a finite alphabet.

(e) The language $\{< D_1, D_2 > |$the DFAs D_1 and D_2 recognize the same language$\}$ is Turing decidable.

(f) If $L_1 \leq_T L_2$ i.e., if L_1 Turing reduces to L_2 , then if L_1 is Turing decidable then so is L_2.

Q4. $(2 \times 4 = 8)$

(a) Describe the language Th$(N, +, \times)$.

(b) Give a sentence in Th$(N, +, \times)$ that expresses the commutative law of multiplication. You may assume that in addition to the usual logic operators you also have the implication operator ($\Rightarrow$), ie if ϕ_1 and ϕ_2 are formulas then $\phi_1 \Rightarrow \phi_2$ is a formula.

(c) State formally what it means for a proof to be verifiable and a proof system to be sound.

(d) Given that Th$(N, +, \times)$ is not decidable, and that provable statements in Th$(N, +, \times)$ are Turing recognizable, show that there is at least one statement in the theory that is not provable.

Q5. $(2 \times 5 = 10)$

(a) Define the complexity classes P and NP.

(b) What is meant by a polynomial time Turing reduction $L_1 \leq_P L_2$?

(c) Define the class of *NP Complete* problems?

(d) The language CLIQUE contains all pairs $< G, k >$ where the graph G has k vertices which form a complete subgraph. The language INDEPENDENT-SET contains all pairs of the form $< G, k >$ where the graph G has k vertices which have no edges between them. Use the fact that CLIQUE is *NP-Complete* to show that INDEPENDENT-SET is *NP-Complete*.

(e) CLIQUE $\in$ *NP*. So both CLIQUE and its compliment have a polynomial time verifiable certificate. True or False? Justify.